Spherical Spline Approximation and its Application to Radio Occultation Data for Climate Monitoring

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Spherical Spline Approximation

Temperature measurements from Champ, 08.06.2008, 137 data points
Temperature measurements from Champ, 08.06.2008, 137 data points

Approximation of 137 data points, $\delta = 0$
Part 1: Spherical Spline Approximation
Spherical Spline Approximation

Some Definitions

\[ \Omega = \{ \xi \in \mathbb{R}^3 \mid \| \xi \| = 1 \} \]

\[ \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \]

\[ \Delta^* = \Delta|_{\Omega} \]
Up until now, spherical splines were defined by a function \( S \) of the form

\[
S(\eta) = \sum_{k=1}^{N} a_k K(\eta, \eta_k),
\]

where \( K(\cdot, \cdot) \) is defined by an infinite bilinear expansion of spherical harmonics \( Y_{n,j} \)

\[
K(\eta, \xi) = \sum_{n=1}^{\infty} \sum_{j=1}^{2n+1} \frac{1}{n(n+1)^2} Y_{n,j}(\eta) Y_{n,j}(\xi).
\]

The infinite bilinear expansion complicates the calculation and evaluation of the spline function defined as above. Hence, we take a new approach in order to define spline functions by means of the second iterated Green’s function on the sphere.
Definition: The function \( G(\Delta^*; \cdot, \cdot) : (\xi, \eta) \mapsto G(\Delta^*; \xi, \eta), -1 \leq \xi \cdot \eta < 1 \) is called Green’s function on \( \Omega = \{ \xi \in \mathbb{R}^3 \mid |\xi| = 1 \} \) with respect to the Beltrami operator \( \Delta^* \), if it satisfies the following properties:

I) (Differential equation) for every fixed \( \xi \in \Omega \), \( \eta \mapsto G(\Delta^*; \xi, \eta) \) is infinitely continuously differentiable on the set \( \{ \eta \in \Omega \mid -1 \leq \xi \cdot \eta < 1 \} \) such that

\[
\Delta^*_\eta G(\Delta^*; \xi, \eta) = -\frac{1}{4\pi} \quad -1 \leq \xi \cdot \eta < 1.
\]

II) (Characteristic singularity) for every \( \xi \in \Omega \), the function

\[
\eta \mapsto G(\Delta^*; \xi, \eta) - \frac{1}{4\pi} \ln(1 - \xi \cdot \eta)
\]

is continuously differentiable on \( \Omega \).
III) (Rotational symmetry) for all orthogonal transformations $A$ the following equation holds:

$$G(\Delta^*; A\xi, A\eta) = G(\Delta^*; \xi, \eta)$$

IV) (Normalization) for every $\xi \in \Omega$ we have

$$\int_{\Omega} G(\Delta^*; \xi, \eta) d\omega(\eta) = 0.$$
Properties of Green’s Function with Respect to $\Delta^*$:

I) The function $G(\Delta^*; \xi, \eta)$ is uniquely determined by its defining properties i) - iv).

II) Green’s function $G(\Delta^*; \xi, \eta)$ has the bilinear expansion

$$G(\Delta^*; \xi, \eta) = -\frac{1}{4\pi} \sum_{m=1}^{\infty} \frac{2m+1}{\lambda_m} P_m(\xi \cdot \eta), \quad -1 \leq \xi \cdot \eta < 1.$$  

III) For $\xi, \eta \in \Omega$ with $-1 \leq \xi \cdot \eta < 1$ Green’s function with respect to the Beltrami operator $\Delta^*$ has the following expression:

$$G(\Delta^*; \xi, \eta) = \frac{1}{4\pi} \ln(1 - \xi \cdot \eta) + \frac{1}{4\pi} - \frac{1}{4\pi} \ln(2)$$
Definition: Let \( G^{(2)}(\Delta^*; \xi, \eta) \) be defined by

\[
G^{(2)}(\Delta^*; \xi, \eta) = \int_\Omega G(\Delta^*; \xi, \zeta)G(\Delta^*; \zeta, \eta)d\omega(\zeta).
\]

The function \( G^{(2)}(\Delta^*; \xi, \eta) \) is called second iterated Green’s function with respect to the Beltrami operator \( \Delta^* \).

It can be shown, that the second iterated Green function has the bilinear expansion

\[
G^{(2)}(\Delta^*; \xi, \eta) = \frac{1}{4\pi} \sum_{m=1}^{\infty} \frac{2m + 1}{\lambda_m^2} P_m(\xi \cdot \eta), \quad -1 \leq \xi \cdot \eta \leq 1
\]
I) For the second iterated Green function corresponding to the Beltrami operator $\Delta^*$ it follows:

$$\Delta^*_\eta G^{(2)}(\Delta^*; \xi, \eta) = G(\Delta^*; \xi, \eta)$$

II) The second iterated Green's function corresponding to the Beltrami operator $\Delta^*$ is continuous and has the expression:

$$G^{(2)}(\Delta^*; \xi, \eta) = \begin{cases} 
\frac{1}{4\pi}, & 1 - \xi \cdot \eta = 0 \\
\frac{1}{4\pi} (1 - \ln(1 - \xi \cdot \eta)(\ln(1 + \xi \cdot \eta) - \ln(2)) \\
- \mathcal{L}_2(1 - \frac{t}{2}) - (\ln(2))^2 + \ln(2)\ln(1 + \xi \cdot \eta)), & 1 \pm \xi \cdot \eta \neq 0 \\
\frac{1}{4\pi} - \frac{\pi}{24}, & 1 + \xi \cdot \eta = 0
\end{cases}$$

where the function $\mathcal{L}_2$ is called dilogarithm and is defined as

$$\mathcal{L}_2(x) = - \int_0^x \frac{\ln(1 - t)}{t} dt = \sum_{k=1}^{\infty} \frac{x^k}{k^2}.$$
Definition: Let the $N$ points $\eta_1, \ldots, \eta_N$ be a fundamental system of order 0 on the unit sphere $\Omega$. Then the function

$$S(\eta) = c + \sum_{k=1}^{N} a_k G^{(2)}(\Delta^*; \eta, \eta_k), \quad \eta \in \Omega, \ c = \text{const}$$

(1)

is called natural spherical spline function of order 0 corresponding to the nodes $\eta_1, \ldots, \eta_N$, if the vector $a = (a_1, \ldots, a_N)^T$ satisfies the linear equation system $Aa = 0$, where

$$A = (1, \ldots, 1).$$

The class of all natural spherical spline functions of order 0 corresponding to the nodes $\eta_1, \ldots, \eta_N$ is denoted by $S(\eta_1, \ldots, \eta_N)$. Further on, let $y = (y_1, \ldots, y_N)$ be an arbitrary $\mathbb{R}$-vector. Then, there exists a unique spline $S \in S(\eta_1, \ldots, \eta_N)$, such that

$$S(\eta_k) = y_k.$$
In order to determine the constants $a_k$ and $c$, the linear equation system

$$\begin{pmatrix} G & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix}$$

has to be solved. Here, $G$ is defined as

$$G = \begin{pmatrix} G^{(2)}(\Delta^*; \eta_1, \eta_1) & \cdots & G^{(2)}(\Delta^*; \eta_1, \eta_N) \\ \vdots & \ddots & \vdots \\ G^{(2)}(\Delta^*; \eta_N, \eta_1) & \cdots & G^{(2)}(\Delta^*; \eta_N, \eta_N) \end{pmatrix}.$$ 

The unique solution of the linear equation system is given by

$$a = G^{-1} y - G^{-1} A^T c \quad \text{with} \quad c = (A G^{-1} A^T)^{-1} A G^{-1} y,$$
Let \((\eta_1, y_1), \ldots, (\eta_N, y_N)\) be \(N\) data points, where \(\eta_1, \ldots, \eta_N\) is a fundamental system of order 0 on \(\Omega\). Let \(S_N \in S(\eta_1, \ldots, \eta_N)\) be the unique natural spline which interpolates the data points \(y_1, \ldots, y_N\). Then, for all twice continuously differentiable functions \(F\) on \(\Omega\), which interpolate the data points \(y_1, \ldots, y_N\), the following equation holds true:

\[
\int_{\Omega} (\Delta^*_\eta S_N(\eta))^2 d\omega(\eta) \leq \int_{\Omega} (\Delta^*_\eta F(\eta))^2 d\omega(\eta)
\]
The problem of fitting a smooth function to a given dataset \((\eta_1, y_1), \ldots, (\eta_N, y_N)\) is given by finding a function \(F\), such that the functional

\[
\sigma_{\beta,\delta}(F) = \sum_{k=1}^{N} \left( \frac{F(\eta_k) - y_k}{\beta_k} \right)^2 + \delta \int_{\Omega} (\Delta^*_\eta F(\eta))^2 d\omega(\eta)
\]

is minimized in \(H^{(2)}(\Omega)\), where \(\beta_k\) are given positive weights and \(\delta \geq 0\) an arbitrary parameter, which gives a measure for the desired smoothness.
Let $\delta, \beta_1, \ldots, \beta_N$ be given positive constants and $(\eta_k, y_k), \ 1 \leq k \leq N$ be given data points. Then there exists a unique spline function $S \in S(\eta_1, \ldots, \eta_N)$ such that the inequality

$$\sigma_{\beta, \delta}(S) \leq \sigma_{\beta, \delta}(F)$$

is valid for all $F \in H^2(\Omega)$ with equality only if $F = S$. Further on, if $S$ is given by Equation (1), then $S$ is uniquely determined by the equation system

$$S(\eta_k) + \delta \beta_k^2 a_k = y_k \quad (k = 1, \ldots, N).$$

The linear equation system can be written as

$$
\begin{pmatrix}
G + \delta B & A^T \\
A & 0
\end{pmatrix}
\begin{pmatrix}
a \\
A c
\end{pmatrix}
=
\begin{pmatrix}
y \\
0
\end{pmatrix},
$$

where $B = \begin{pmatrix}
\beta_1^2 & 0 \\
& \ddots \\
0 & \beta_N^2
\end{pmatrix}$.
Part 2: Application
Temperature Profile in Estes Park, Summer 2008

Mean temperature profile in Estes Park, summer 2008

- Height in km
- Temperature in °C
Application

Smoothing

Approximation of 137 data points, $\delta=0$

Temperature measurements from Champ, 08.06.2008, 137 data points
Approximation of 137 data points, $\delta=0$

Approximation of 137 data points with 10% white noise, $\delta=0.0004$

Difference between the approximation of the white noise data with $\delta=0.0004$ and $\delta=0$
Approximation of 137 data points, $\delta=0$

Approximation of 137 data points with 10% white noise, $\delta=0.0004$

Difference between the approximation of the white noise data with $\delta=0.0004$ and $\delta=0$

Difference between the approximation with and without white noise
Thank you!