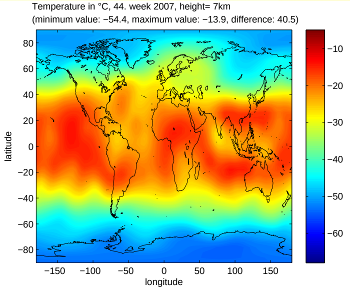
TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

Spherical Spline Approximation and its Application to Radio Occultation Data for Climate Monitoring

2nd International Radio Occultation Workshop 2012



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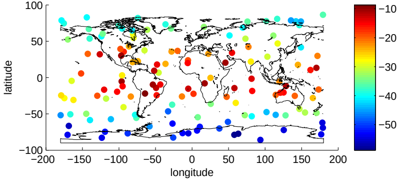
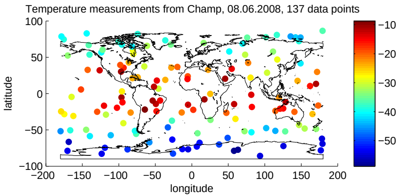
RP cluster of excellence

GEOMATHEMATIK (CM)²

Temperature measurements from Champ, 08.06.2008, 137 data points




Temperature measurements from Champ, 08.06.2008, 137 data points
latitude
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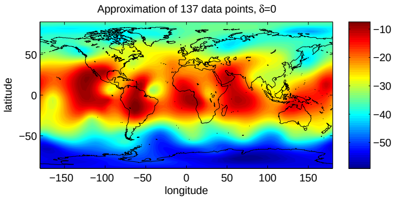
Approximation of 137 data points, δ=0
latitude
longitude
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Part 1: Spherical Spline Approximation

$$\Omega = \{\xi \in \mathbb{R}^3 \mid \|\xi\| = 1\}$$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\Delta^* = \Delta|_\Omega$$

Up until now, spherical splines were defined by a function S of the form

$$S(\eta) = \sum_{k=1}^{N} a_k K(\eta, \eta_k),$$

where $K(\cdot,\cdot)$ is defined by an infinite bilinear expansion of spherical harmonics $Y_{n,j}$

$$K(\eta, \xi) = \sum_{n=1}^{\infty} \sum_{j=1}^{2n+1} \frac{1}{(n(n+1))^2} Y_{n,j}(\eta) Y_{n,j}(\xi).$$

The infinite bilinear expansion complicates the calculation and evaluation of the spline function defined as above. Hence, we take a new approach in order to define spline functions by means of the second iterated Green's function on the sphere.

Definition: The Function $G(\Delta^*;\cdot,\cdot):(\xi,\eta)\longmapsto G(\Delta^*;\xi,\eta)$, $-1\le\xi\cdot\eta<1$ is called Green's function on $\Omega=\{\xi\in\mathbb{R}^3\mid|\xi|=1\}$ with respect to the Beltrami operator Δ^*, if it satisfies the following properties:

I) (Differential equation) for every fixed $\xi\in\Omega$, $\eta\longmapsto G(\Delta^*;\xi,\eta)$ is infinitely continuously differentiable on the set $\{\eta\in\Omega\mid -1\le\xi\cdot\eta<1\}$ such that

$$\Delta^*_\eta G(\Delta^*;\xi,\eta)=-\frac{1}{4\pi}\qquad -1\le\xi\cdot\eta<1.$$

II) (Characteristic singularity) for every $\xi\in\Omega$, the function

$$\eta\longmapsto G(\Delta^*;\xi,\eta)-\frac{1}{4\pi}\ln(1-\xi\cdot\eta)$$

is continuously differentiable on Ω.

III) (Rotational symmetry) for all orthogonal transformations **A** the following equation holds:

$$G(\Delta^*; \mathbf{A}\xi, \mathbf{A}\eta) = G(\Delta^*; \xi, \eta)$$

IV) (Normalization) for every $\xi \in \Omega$ we have

$$\int_\Omega G(\Delta^*; \xi, \eta) d\omega(\eta) = 0.$$

Properties of Green's Function with Respect to Δ^*:

I) The function $G(\Delta^*;\xi,\eta)$ is uniquely determined by its defining properties i) - iv).

II) Green's function $G(\Delta^*;\xi,\eta)$ has the bilinear expansion

$$G(\Delta^*;\xi,\eta) = -\frac{1}{4\pi}\sum_{m=1}^{\infty}\frac{2m+1}{\lambda_m}P_m(\xi\cdot\eta), \qquad -1 \leq \xi\cdot\eta < 1.$$

III) For $\xi,\eta \in \Omega$ with $-1 \leq \xi\cdot\eta < 1$ Green's function with respect to the Beltrami operator Δ^* has the following expression:

$$G(\Delta^*;\xi,\eta) = \frac{1}{4\pi}\ln(1-\xi\cdot\eta) + \frac{1}{4\pi} - \frac{1}{4\pi}\ln(2)$$

Definition: Let $G^{(2)}(\Delta^*;\xi,\eta)$ be defined by

$$G^{(2)}(\Delta^*;\xi,\eta) = \int_\Omega G(\Delta^*;\xi,\zeta)G(\Delta^*;\zeta,\eta)d\omega(\zeta).$$

The function $G^{(2)}(\Delta^*;\xi,\eta)$ is called second iterated Green's function with respect to the Beltrami operator Δ^*.

It can be shown, that the second iterated Green function has the bilinear expansion

$$G^{(2)}(\Delta^*;\xi,\eta) = \frac{1}{4\pi}\sum_{m=1}^{\infty}\frac{2m+1}{\lambda_m^2}P_m(\xi\cdot\eta), \qquad -1 \leq \xi\cdot\eta \leq 1$$

I) For the second iterated Green function corresponding to the Beltrami operator Δ^* it follows:

$$\Delta^*_\eta G^{(2)}(\Delta^*;\xi,\eta) = G(\Delta^*;\xi,\eta)$$

II) The second iterated Green's function corresponding to the Beltrami operator Δ^* is continuous and has the expression:

$$G^{(2)}(\Delta^*;\xi,\eta) = \begin{cases} \frac{1}{4\pi} & , \quad 1-\xi\cdot\eta = 0 \\ \frac{1}{4\pi}(1 - \ln(1-\xi\cdot\eta)(\ln(1+\xi\cdot\eta) - \ln(2)) \\ -\mathfrak{L}_2(\frac{1-t}{2}) - (\ln(2))^2 + \ln(2)\ln(1+\xi\cdot\eta)) & , \quad 1\pm\xi\cdot\eta \neq 0 \\ \frac{1}{4\pi} - \frac{\pi}{24} & , \quad 1+\xi\cdot\eta = 0 \end{cases}$$

where the function $\mathfrak{L}_2$ is called dilogarithm and is defined as

$$\mathfrak{L}_2(x) = -\int_0^x \frac{\ln(1-t)}{t}\,dt = \sum_{k=1}^{\infty} \frac{x^k}{k^2}.$$

Definition: Let the N points $\eta_1, \ldots, \eta_N$ be a fundamental system of order 0 on the unit sphere Ω. Then the function

$$S(\eta) = c + \sum_{k=1}^{N} a_k G^{(2)}(\Delta^*; \eta, \eta_k), \qquad \eta \in \Omega, \; c = const \tag{1}$$

is called natural spherical spline function of order 0 corresponding to the nodes $\eta_1, \ldots, \eta_N$, if the vector $a = (a_1, \ldots, a_N)^T$ satisfies the linear equation system $Aa = 0$, where

$$A = (1, \ldots, 1)\,.$$

The class of all natural spherical spline functions of order 0 corresponding to the nodes $\eta_1, \ldots, \eta_N$ is denoted by $\mathcal{S}(\eta_1, \ldots, \eta_N)$.
Further on, let $y = (y_1, \ldots, y_N)$ be an arbitrary $\mathbb{R}$-vector. Then, there exists a unique spline $S \in \mathcal{S}(\eta_1, \ldots, \eta_N)$, such that

$$S(\eta_k) = y_k\,.$$

In order to determine the constances a_k and c, the linear equation system

$$\begin{pmatrix} G & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix}$$

has to be solved. Here, G is defined as

$$G = \begin{pmatrix} G^{(2)}(\Delta^*; \eta_1, \eta_1) & \cdots & G^{(2)}(\Delta^*; \eta_1, \eta_N) \\ \vdots & & \vdots \\ G^{(2)}(\Delta^*; \eta_N, \eta_1) & \cdots & G^{(2)}(\Delta^*; \eta_N, \eta_N) \end{pmatrix}.$$

The unique solution of the linear equation system is given by

$$a = G^{-1}y - G^{-1}A^T c \qquad \text{with } c = (AG^{-1}A^T)^{-1}AG^{-1}y,$$

Let $(\eta_1, y_1), \ldots, (\eta_N, y_N)$ be N data points, where $\eta_1, \ldots, \eta_N$ is a fundamental system of order 0 on Ω. Let $S_N \in \mathcal{S}(\eta_1, \ldots, \eta_N)$ be the unique natural spline which interpolates the data points $y_1, \ldots, y_N$. Then, for all twice continuously differentiable functions F on Ω, which interpolate the data points $y_1, \ldots, y_N$, the following equation holds true:

$$\int_\Omega (\Delta_\eta^* S_N(\eta))^2 d\omega(\eta) \leq \int_\Omega (\Delta_\eta^* F(\eta))^2 d\omega(\eta)$$

The problem of fitting a smooth function to a given dataset $(\eta_1, y_1), \ldots, (\eta_N, y_N)$ is given by finding a function F, such that the functional

$$\sigma_{\beta,\delta}(F) = \sum_{k=1}^{N} \left(\frac{F(\eta_k) - y_k}{\beta_k} \right)^2 + \delta \int_{\Omega} (\Delta^*_\eta F(\eta))^2 d\omega(\eta)$$

is minimized in $H^{(2)}(\Omega)$, where β_k are given positive weights and $\delta \geq 0$ an arbitrary parameter, which gives a measure for the desired smoothness.

Let δ, $\beta_1, \ldots, \beta_N$ be given positive constants and (η_k, y_k), $1 \leq k \leq N$ be given data points. Then there exists a unique spline function $S \in \mathcal{S}(\eta_1, \ldots, \eta_N)$ such that the inequality

$$\sigma_{\beta,\delta}(S) \leq \sigma_{\beta,\delta}(F)$$

is valid for all $F \in H^{(2)}(\Omega)$ with equality only if $F = S$. Further on, if S is given by Equation (1), then S is uniquely determined by the equation system

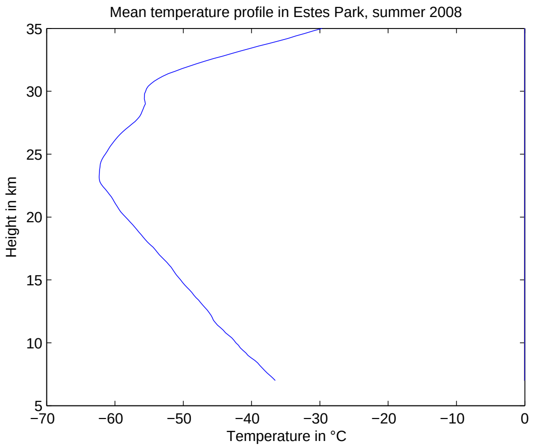
$$S(\eta_k) + \delta\beta_k^2 a_k = y_k \qquad (k = 1, \ldots, N).$$

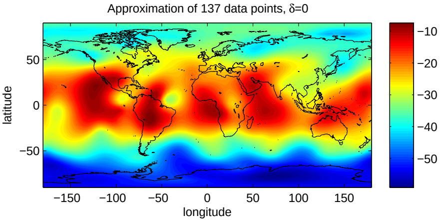
The linear equation system can be written as

$$\begin{pmatrix} G + \delta B & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix}, \text{ where } B = \begin{pmatrix} \beta_1^2 & & 0 \\ & \ddots & \\ 0 & & \beta_N^2 \end{pmatrix}$$

Part 2: Application

Temperature Profile in Estes Park, Summer 2008




Approximation of 137 data points, δ=0
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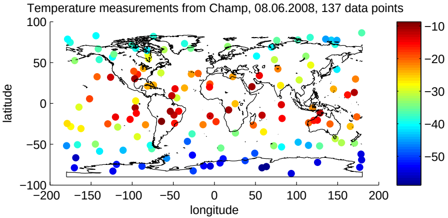
Temperature measurements from Champ, 08.06.2008, 137 data points
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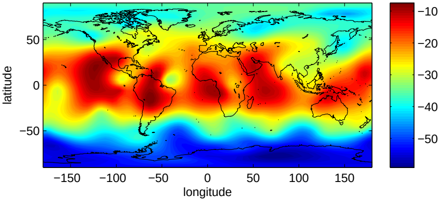
Approximation of 137 data points, δ=0
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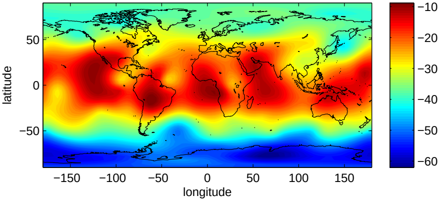
Approximation of 137 data points with 10% white noise, δ=0.0004
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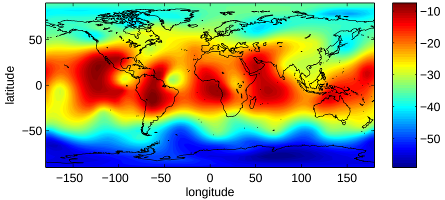
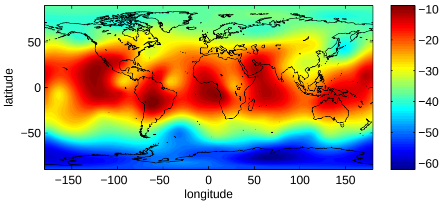
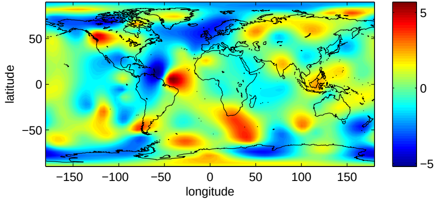
Approximation of 137 data points, δ=0



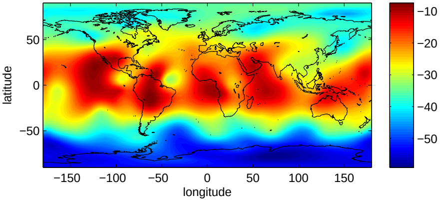
Approximation of 137 data points with 10% white noise, δ=0.0004



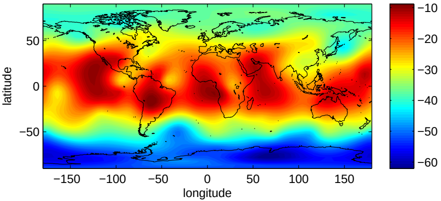
Difference between the approximation of the white noise data with δ=0.0004 and δ=0



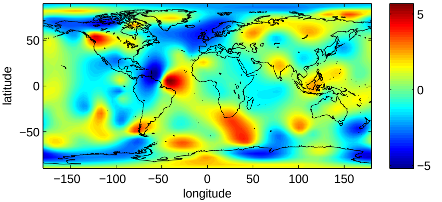
Approximation of 137 data points, δ=0



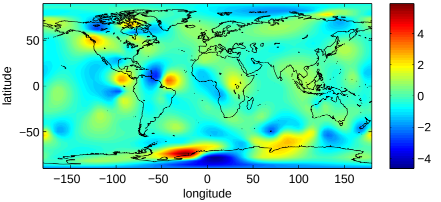
Approximation of 137 data points with 10% white noise, δ=0.0004



Difference between the approximation of the white noise data with δ=0.0004 and δ=0



Difference between the approximation with and without white noise



Thank you!